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VIBRATION DAMPING THROUGH THE USE OF MATERIALS WITH MEMORY†

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Abstract—We study the dynamics of a lumped mass linear oscillator that is damped through the use of a *material with memory* in which the internal dissipative forces depend not only on current but also on previous deformations. This effective memory is governed by two parameters: the relaxation modulus G_0, and relaxation time γ, which also govern the vibration-damping properties of the material. Conditions for optimal damping in the unforced case corresponding to critical damping of a linear oscillator with viscous damping are derived, and the response of the oscillator in the case of sinusoidal excitation is studied. When the relaxation time is small the history type damping is modeled approximately by the action of a classical viscous damper with small viscosity. However, when the relaxation time is sufficiently large, this damping mechanism adds to the system a new higher resonance frequency that depends on G_0 and γ. Since the oscillator is active over a wide range of frequencies, it has potential applications to the development of adaptive damping devices. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Viscoelastic materials are widely used as elements in mechanisms of vibration control most importantly due to their damping effects. Such applications are found, for example, in vibration-damping of flexible structures such as beams, plates, shells etc., where the viscoelastic material is made to vibrate with the structural member for example in the form of a layer attached to a beam or a plate. The use of such layers in the control of vibrations of spherical shells, cylindrical shells, etc. has been studied by several authors, e.g. Okazaki *et al.* (1990), Gautham *et al.* (1994), Culkowski *et al.* (1971). The case of nonlinear radial vibrations of a viscoelastic spherical shell capable of undergoing a phase transformation is studied by Fosdick *et al.* (1997); here the elastic contribution is characterized by a non-convex strain energy function.

A simpler but more wide-spread use of the effects of viscosity is found in viscous dampers. These are important components of many mechanical applications where vibrations need to be damped, isolated or controlled in some other manner. The intrinsic mechanism of damping in such "dashpot" devices is the action of a velocity dependent force that is invariably in a direction opposing the velocity of the vibrating mass, and to first order proportional to the magnitude of the velocity.

An alternative way of characterizing this force, that has the advantage of being generalized to a larger class of dissipative phenomena, is that it is only indirectly dependent on the velocity; its direct dependence being on the position that the object occupied (in relation to its present position), a period of time Δt before the current time. These types of history dependent dissipative forces are supplied by materials such as gum rubber and high polymer solutions for which the stresses in the material depend on past as well as present states of deformation. Such materials are termed *history type materials*, or materials with memory (see, e.g., Truesdell and Noll (1965)). If, in addition, the relative deformation of the recent past is more important in determining the force than that further back in time, the material is said to have "fading memory". A rigorous treatise on the thermodynamics of such materials has been given by Coleman (1964).

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The history type modeling of viscoelastic materials was first proposed early on by Boltzmann in 1874. In the simplest situation, the dissipative nature of materials with memory is characterized by two material parameters: a relaxation time γ, which determines the rate at which the influence of past states of strain on present stress diminishes with elasped time, and a relaxation modulus G_0, which determines the overall strength of the history dependence of the stress. More generally, history type materials may exhibit a spectrum of relaxation times, all influencing the behavior of the material simultaneously, and, depending on material symmetry, a number of independent relaxation moduli, but for simplicity, we shall only consider one dimensional motions and a single relaxation modulus. In some cases, the parameters γ and G_0 themselves may be sensitive to outside effects such as temperature or electric field in electro-rheological materials, and this suggests the possibility of tuning them to desired values in a specific application or using them as interactive modulating devices.

The aim of this paper is to investigate the possible use of viscoelastic materials with memory in the practical context of the damping of oscillators and to compare this type of damping to the conventional type of viscous damping. We study the interplay of the relaxation modulus and the relaxation time in determining the nature of the damping in both the cases of forced and free oscillations. Of close relevance to this work is a study of the stability and nonlinear oscillations of cylindrical and spherical viscoelastic shells by Fosdick and Yu (1996).

We begin in Section 2 by introducing a model of the oscillator we wish to consider. We give a brief summary of some of the main elements and assumptions from continuum mechanics on which the remainder of our work is based and we obtain the governing equations of motion for the oscillator. It is shown how these may be expressed as three first order nonlinear ordinary differential equations.

A linearized version of the governing equations is given in Section 3. It is shown that the same linear dynamical system may be obtained through the use of a discrete mechanical model for the viscoelastic material. A study of the linear system shows that the history dependence leads to important qualitative differences from the case of classical viscous damping. For example, we show that the resonance frequency can be significantly altered by changing the relaxation time of the material.

In Section 4, we consider the limiting cases of "small" and "large" relaxation times. In each of these cases the governing differential equations may be approximated by second order systems. In the first case, the memory of the material is so short that it is adequately described by the action of a classical viscosity. In the second case the material has long memory and it behaves as if it was purely elastic.

2. CONSTITUTIVE MODEL AND DYNAMICAL EQUATIONS

In the main body of this work we shall study the motion of a one degree of freedom oscillator with a lumped mass m which is subject to a force $p(t)$ and a restoring force due to the action of a viscoelastic material with memory. Eventually, we assume that the motion and all forces are uniaxial, as indicated in Fig. 1, and we shall suppose that the viscoelastic damper has a relatively small mass when compared to m.

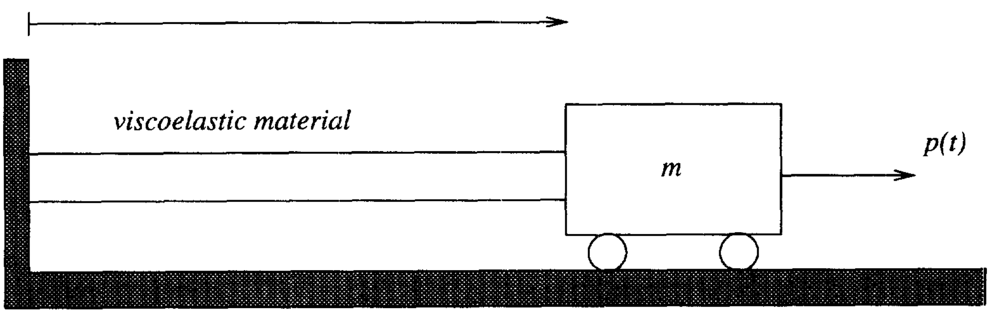

Fig. 1. Schematic diagram of a linear oscillator with history type force.

When a solid viscoelastic body is subject to a deformation history, it is common to assume that the Cauchy stress $\mathbf{T}$ at each particle $\mathbf{X}$ of the body and at the current time t is determined by the history of the deformation gradient for that particle. If we let $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ denote the motion of the body then the deformation gradient history is given by $\mathbf{F}(\mathbf{X}, t-s) \equiv \nabla\boldsymbol{\chi}(\mathbf{X}, t-s) \forall s \geqslant 0$. The class of isotropic materials known as finite-linear viscoelastic solids (see, e.g., Coleman and Noll (1961)), is characterized by the constitutive assumption

$$\mathbf{T}(\mathbf{X}, t) = \bar{\mathbf{T}}^e(\mathbf{F}(\mathbf{X}, t)) + \int_0^\infty \{\lambda(\mathbf{F}(\mathbf{X}, t), s)\mathbf{1}\, tr\mathbf{J}_t(\mathbf{X}, t-s) + 2\mu(\mathbf{F}(\mathbf{X}, t), s)\mathbf{J}_t(\mathbf{X}, t-s)\}\, \mathrm{d}s, \tag{1}$$

where the elastic part $\bar{\mathbf{T}}^e(\cdot)$ and the relaxation-like scalar functions $\lambda(\cdot, s)$ and $\mu(\cdot, s)$ are isotropic functions of the left Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, and where

$$\mathbf{J}_t(\mathbf{X}, \tau) = \mathbf{C}_t(\mathbf{X}, \tau) - \mathbf{1}, \tag{2}$$

with the relative right Cauchy–Green strain tensor $\mathbf{C}_t(\mathbf{X}, \tau)$ computed according to

$$\mathbf{C}_t(\mathbf{X}, \tau) = \mathbf{F}^{-1^{\mathrm{T}}}(\mathbf{X}, t)\mathbf{F}^{\mathrm{T}}(\mathbf{X}, \tau)\mathbf{F}(\mathbf{X}, \tau)\mathbf{F}^{-1}(\mathbf{X}, t). \tag{3}$$

The main features of this constitutive theory are that it is a possible exact theory of material behavior, and the presence of the deformation gradient history is included in an isotropic linear history functional through a properly invariant nonlinear measure of strain history, i.e., $\mathbf{J}_t(\mathbf{X}, t-s)$ for all $s \geqslant 0$. The constitutive assumption (1), as it stands, is highly nonlinear and it has classical linear viscoelasticity as its first order approximation (see, e.g., Coleman and Noll (1961)).

In order to reduce (1) to a more tractable theory, we shall assume that the relaxation functions $\lambda(\mathbf{F}, s)$ and $\mu(\mathbf{F}, s)$ are independent of $\mathbf{F}$. Moreover, in the one-dimensional setting suggested by Fig. 1, we take the motion of the viscoelastic body to be described by the scalar equation $x = \chi(\mathbf{X}, t)$, and by analogy to (1) we assume that the axial force $f(X, t)$ on X at time t is given by

$$f(X, t) = \bar{f}^e(F(X, t)) + \int_0^\infty \varphi(s) J_t(X, t-s)\, \mathrm{d}s. \tag{4}$$

Here, $F(X, t) = \partial\chi(X, t)/\partial X$, and from (2) and (3) we have

$$J_t(X, t-s) = \frac{[F(X, t-s)]^2 - [F(X, t)]^2}{[F(X, t)]^2}. \tag{5}$$

In addition, we shall introduce the simple relaxation function

$$G(s) \equiv G_0 e^{-s/\gamma}, \tag{6}$$

where $G_0 > 0$ and $\gamma > 0$, and take

$$\varphi(s) = \dot{G}(s) \tag{7}$$

in (4).†

The engineering stress (Piola–Kirchhoff) is given by $f(X, t)/A_0$, where A_0 is the (constant) referential cross section area of the bar. Thus, the balance of momentum for the bar is given by

$$\frac{1}{A_0}\frac{\partial f(X,t)}{\partial X} = \rho_0 \ddot{\chi}(X,t), \tag{8}$$

where ρ_0 is the constant reference mass density. If we let L_0 denote the referential length of the viscoelastic bar, then the dynamical equation for the mass m is

$$m\ddot{\chi}(L_0, t) = p(t) - f(L_0, t), \tag{9}$$

and the governing system of dynamical equations is complicated by the partial differential character of (8). However, in the limiting case when the inertia of the bar can be neglected, it is clear that (8) requires only that the axial force $f(X, t)$ be independent of X. If we suppose that the deformation of the bar is homogeneous, which seems reasonable if $\bar{f}^e(\cdot)$ is monotone increasing, so that

$$x = \chi(X, t) = X\bar{x}(t), \tag{10}$$

then $F(X, \tau) = \bar{x}(\tau)$ and (5), (6), and (7) show that $f(X, t) = f(t)$, where

$$f(t) = \bar{f}^e(\bar{x}(t)) + \int_0^\infty \dot{G}(s)\frac{\bar{x}^2(t-s) - \bar{x}^2(t)}{\bar{x}^2(t)}\,\mathrm{d}s. \tag{11}$$

Then, the single dynamical equation for determining the motion of the mass m is

$$mL_0\ddot{\bar{x}}(t) = p(t) - f(t), \tag{12}$$

which, by (11), is a nonlinear integral-differential equation for determining $\bar{x}(\cdot)$. If we consider a special forcing function $p(t) = P\cos\Omega t$, then by (6), (11), and (12) we find that

$$mL_0\ddot{\bar{x}}(t) = -\bar{f}^e(\bar{x}(t)) + \frac{G_0}{\gamma}\int_0^\infty e^{-s/\gamma}\frac{\bar{x}^2(t-s) - \bar{x}^2(t)}{\bar{x}^2(t)}\,\mathrm{d}s + P\cos\Omega t. \tag{13}$$

Here, it is convenient to rewrite this as a system of first order ordinary differential equations by defining the auxiliary function

†In a one-dimensional relaxation experiment, an undistorted and unloaded filament is subjected to a homogeneous step extensional strain ε at time $t = 0$ and thereafter held fixed. There is an observed corresponding initial axial force which decays asymptotically to a constant lower value for long time. Using the model of (4)–(7), and the common identification $\varepsilon \equiv \mathrm{F} - 1$, we readily find that the force is homogeneous (i.e., independent of X) and decays according to

$$f(t) = \bar{f}^e(1+\varepsilon) + G_0\frac{(1+\varepsilon)^2 - 1}{(1+\varepsilon)^2}e^{-t/\gamma}.$$

Thus, the long-time (i.e., $t \to \infty$) and initial (i.e., $t = 0$) force-strain responses are elastic and in general nonlinear functions of strain. The effect of the viscoelastic relaxation for this model is exhibited in the second term above, which contains a single relaxation time constant γ and which depends explicitly upon the strain. While there are materials which exhibit this *general* behavior, single relaxation times are not expected to completely model the detailed relaxation response. Our aim in this work is to use this model solely to illustrate certain phenomenological possibilities associated with nonlinearities and relaxation.

$$\tilde{\zeta}(t) = \int_0^\infty e^{-s/\gamma} \frac{\tilde{x}^2(t-s) - \tilde{x}^2(t)}{\tilde{x}^2(t)} \, \mathrm{d}s. \tag{14}$$

Then, it readily follows that (13) has the equivalent form

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{\eta}(t), \\ mL_0\dot{\tilde{\eta}}(t) &= -\bar{f}^e(\tilde{x}(t)) + \frac{G_0}{\gamma}\tilde{\zeta}(t) + P\cos\Omega t, \\ \dot{\tilde{\zeta}}(t) &= -\left[\frac{1}{\gamma} + \frac{2\tilde{\eta}(t)}{\tilde{x}(t)}\right]\tilde{\zeta}(t) - \gamma\frac{2\tilde{\eta}(t)}{\tilde{x}(t)}. \end{aligned} \tag{15}$$

Clearly, an equilibrium point $(\tilde{x}^*, \tilde{\eta}^*, \tilde{\zeta}^*)$ for (15) (when $P = 0$) is given by $(\tilde{x}, \tilde{\eta}, \tilde{\zeta}) = (\tilde{x}^*, 0, 0)$, where $\tilde{x}^*$ is any root of the equation $\bar{f}^e(\tilde{x}^*) = 0$. In what follow we assume $\tilde{x}^* = 1$, i.e., the motion takes place about the undistorted state.

3. THE LINEAR SYSTEM

We consider, now, those motions that are sufficiently well described by a linearized form of (15). To this end, and noting that such will occur in the vicinity of the equilibrium points, we let $\tilde{\xi}(t)$ be such that

$$\tilde{x}(t) = \tilde{x}^* + \tilde{\xi}(t) \tag{16}$$

and observe that the linearized form of (15) is

$$\begin{aligned} \dot{\tilde{\xi}}(t) &= \tilde{\eta}(t), \\ mL_0\dot{\tilde{\eta}} &= -k\tilde{\xi}(t) + \frac{G_0}{\gamma}\tilde{\zeta}(t) + P\cos\Omega t, \\ \dot{\tilde{\zeta}}(t) &= -\frac{1}{\gamma}\tilde{\zeta}(t) - 2\gamma\tilde{\eta}(t), \end{aligned} \tag{17}$$

where

$$k \equiv \frac{\mathrm{d}\bar{f}^e}{\mathrm{d}\tilde{x}}(\tilde{x}^*) > 0. \tag{18}$$

We now introduce a dimensionless time variable τ according to

$$\tau = \omega_0 t, \tag{19}$$

where $\omega_0 = \sqrt{k/mL_0}$ is chosen to be the frequency of natural vibrations that would occur if $G_0 = 0$ and $P = 0$. Introduce

$$\xi(\tau) = \tilde{\xi}\left(\frac{\tau}{\omega_0}\right), \eta(\tau) = \frac{1}{\omega_0}\tilde{\eta}\left(\frac{\tau}{\omega_0}\right), \zeta(\tau) = \frac{1}{\gamma}\tilde{\zeta}\left(\frac{\tau}{\omega_0}\right). \tag{20}$$

Then, (17) may be written in the form

$$\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau} = \eta(\tau),$$

$$\frac{\mathrm{d}\eta(\tau)}{\mathrm{d}\tau} = -\xi(\tau)+\Phi_0\zeta(\tau)+p\cos\omega\tau,$$

$$\frac{\mathrm{d}\zeta(\tau)}{\mathrm{d}\tau} = -\frac{1}{\gamma\omega_0}\zeta(\tau)-2\eta(\tau), \tag{21}$$

where

$$\Phi_0 = \frac{G_0}{k}, \tag{22}$$

and

$$p = \frac{P}{k}, \omega = \frac{\Omega}{\omega_0}. \tag{23}$$

3.1. *Discrete mechanical model for the linear viscoelastic oscillator*

It is common to study the dynamical behavior of viscoelastic materials by introducing discrete mechanical spring and dashpot models. Consider, for example, the three-element model of Fig. 2, in which k_0 and k_1 are the linear spring constants and μ is the "viscosity" of the dashpot. The natural lengths of the springs are l_1 and l_0, respectively. Let $x_1(t)$ and $x(t)$ denote the positions at time t as shown in Fig. 2. Then, the restoring force on mass M is

$$F(t) = k_0(x(t)-x_1(t)-l_0) = k_1(x_1(t)-l_1)+\mu\dot{x}_1(t). \tag{24}$$

Clearly, then, we have

$$\dot{x}_1(t)+\frac{k_1+k_0}{\mu}x_1(t) = \frac{k_0}{\mu}(x(t)-l'), \quad l' \equiv \frac{k_0 l_0 - k_1 l_1}{k_0}, \tag{25}$$

which, after integration, yields

$$x_1(t) = \frac{k_0}{\mu}\int_0^\infty e^{-[(k_1+k_0)/\mu]s}(x(t-s)-l')\,\mathrm{d}s. \tag{26}$$

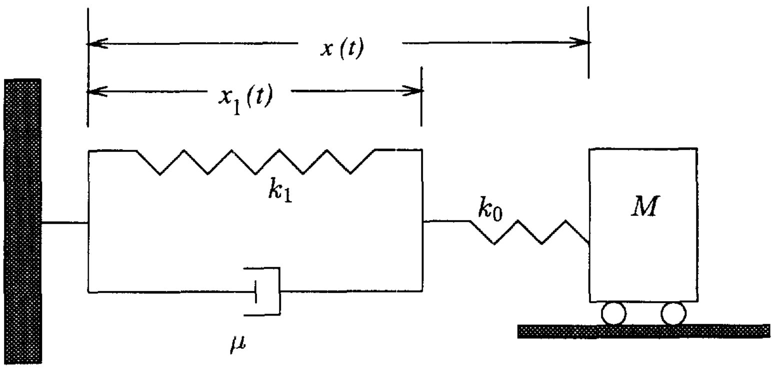

Fig. 2. Discrete mechanical model for the linear viscoelastic oscillator.

Thus, for the total force we have

$$F(t) = k_0(x(t)-l_0) - \frac{k_0^2}{\mu}\int_0^\infty e^{-[(k_1+k_0)/\mu]s}(x(t-s)-l')\,\mathrm{d}s, \tag{27}$$

which has the equivalent form

$$F(t) = \frac{k_1k_0}{k_1+k_0}(x(t)-l_0-l_1) - \frac{k_0^2}{\mu}\int_0^\infty e^{-[(k_1+k_0)/\mu]s}(x(t-s)-x(t))\,\mathrm{d}s. \tag{28}$$

Now, the dynamical equation of free motion for M is given by the integro-differential equation $-F(t) = M\ddot{x}(t)$, which, with the definition

$$z(t) = \int_0^\infty e^{-[(k_1+k_0)/\mu]s}(x(t-s)-x(t))\,\mathrm{d}s, \tag{29}$$

may be written as the system

$$\begin{aligned}
\dot{x}(t) &= y(t),\\
M\dot{y}(t) &= -\frac{k_1k_0}{k_1+k_0}(x(t)-l_0-l_1) + \frac{k_0^2}{\mu}z(t)\\
\dot{z}(t) &= -\frac{k_1+k_0}{\mu}z(t) - \frac{\mu}{k_1+k_0}y(t).
\end{aligned} \tag{30}$$

A comparison of (30) and (17) (with $P = 0$) show that these two systems are equivalent if the following identifications are made:

$$\begin{aligned}
\gamma &= \frac{\mu}{k_1+k_0},\\
\tilde{\xi}(t) &= x(t)-l_0-l_1,\\
\tilde{\eta}(t) &= y(t),\\
\tilde{\zeta}(t) &= 2z(t),\\
mL_0 &= M,\\
k &= \frac{k_0k_1}{k_0+k_1},\\
\frac{2G_0}{\gamma} &= \frac{k_0^2}{\mu}.
\end{aligned} \tag{31}$$

3.2. *The unforced case*

In the case when $p = 0$ the linear system (21) may be written in the matrix form

$$\frac{\mathrm{d}\boldsymbol{\chi}(\tau)}{\mathrm{d}\tau} = \mathbf{A}\boldsymbol{\chi}(\tau), \tag{32}$$

where

$$\boldsymbol{\chi}(\tau) = (\xi, \eta, \zeta)(\tau) \tag{33}$$

and

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & \Phi_0\\ 0 & -2 & -\kappa\end{bmatrix}, \tag{34}$$

and where we have introduced the parameter

$$\kappa=\frac{1}{\gamma\omega_0}>0. \tag{35}$$

The characteristic polynomial for $\mathbf{A}$ is

$$\lambda^3+\kappa\lambda^2+(2\Phi_0+1)\lambda+\kappa=0, \tag{36}$$

which has three roots, one of which is real. The other two roots, depending on the sign of the discriminant D (see the definition below), may be complex conjugate ($D>0$), real and equal ($D=0$), or real and distinct ($D<0$). Nevertheless, all three roots have negative real parts. To see this, first note that the characteristic polynomial of (36) can be rewritten as

$$\lambda^3+\kappa\lambda^2+(2\Phi_0+1)\lambda+\kappa=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3), \tag{37}$$

where λ_i, $i=1,2,3$, are the roots of (36). Thus,

$$\begin{aligned}\lambda_1+\lambda_2+\lambda_3&=-\kappa,\\ \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3&=2\Phi_0+1,\\ \lambda_1\lambda_2\lambda_3&=-\kappa.\end{aligned} \tag{38}$$

Suppose that λ_1 is real and that λ_2 and λ_3 are either real or complex conjugate. Then $(38)_3$ requires that $\lambda_1\lambda_2\lambda_3<0$ and so at least one real root must be negative regardless of whether all are real or not. We may then assume that $\lambda_1<0$ and $\lambda_2\lambda_3>0$ without loss or generality. From $(38)_2$ we readily see that

$$\lambda_2\lambda_3=2\Phi_0+1-\lambda_1(\lambda_2+\lambda_3)$$

and with $(38)_3$ we find

$$\lambda_1\lambda_2\lambda_3=\lambda_1[2\Phi_0+1-\lambda_1(\lambda_2+\lambda_3)]=-\kappa. \tag{39}$$

Finally, eliminating κ between $(38)_1$ and (39), we find

$$\lambda_2+\lambda_3=\frac{2\Phi_0\lambda_1}{1+\lambda_1^2}<0,$$

which implies that $\mathrm{Re}(\lambda_2)=\mathrm{Re}(\lambda_3)<0$ if λ_2 and λ_3 are complex conjugate, and with $\lambda_2\lambda_3>0$ we see that $\lambda_2<0$ and $\lambda_3<0$ if λ_2 and λ_3 are real.

Defining two auxiliary quantities Q and R according to

$$Q=\frac{3(2\Phi_0+1)-\kappa^2}{9} \tag{40}$$

and

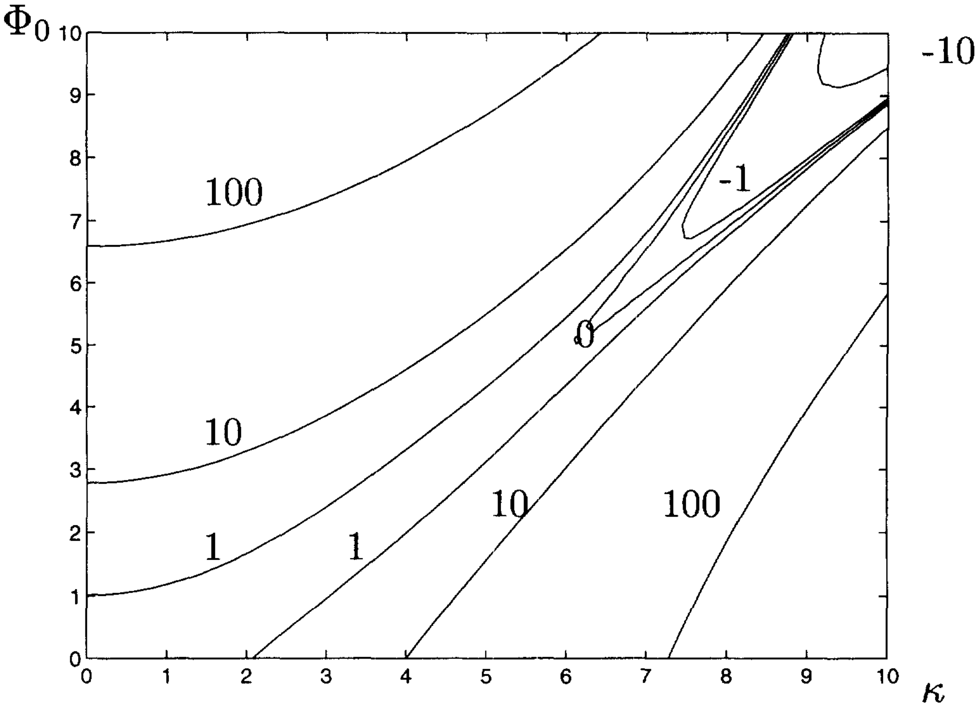

Fig. 3. Contour lines for the discriminant D.

$$R = \frac{9\kappa(2\Phi_0+1) - 27\kappa - 2\kappa^3}{54}, \tag{41}$$

we have for the discriminant D

$$D = Q^3 + R^2. \tag{42}$$

Figure 3 shows how D depends on κ and Φ_0.

An eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_i is easily shown to have the form

$$\mathbf{v}_i = \begin{bmatrix} 1 \\ \lambda_i \\ \dfrac{\lambda_i^2+1}{\Phi_0} \end{bmatrix}. \tag{43}$$

If one assumes that $D > 0$ or $D < 0$, so that $\mathbf{A}$ has three distinct eigenvalues, then there is a coordinate transformation which diagonalizes $\mathbf{A}$. This transformation is affected by a matrix $\mathbf{S}$ whose columns are the eigenvectors of $\mathbf{A}$, i.e.,

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \dfrac{\lambda_1^2+1}{\Phi_0} & \dfrac{\lambda_2^2+1}{\Phi_0} & \dfrac{\lambda_3^2+1}{\Phi_0} \end{bmatrix}. \tag{44}$$

Denoting the new coordinates by $\bar{\chi}(\tau) = (\bar{\xi}, \bar{\eta}, \bar{\zeta})(\tau)$ we have

$$\chi(\tau) = \mathbf{S}\bar{\chi}(\tau) \tag{45}$$

and

$$\frac{\mathrm{d}\bar{\chi}(\tau)}{\mathrm{d}\tau} = \mathbf{\Lambda}\bar{\chi}(\tau), \tag{46}$$

where

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{47}$$

The general solution of (46) may be written in the form

$$\bar{\chi}(\tau) = e^{\Lambda\tau}\bar{\chi}_0 \tag{48}$$

where

$$e^{\Lambda\tau} = \begin{bmatrix} e^{\lambda_1\tau} & 0 & 0 \\ 0 & e^{\lambda_2\tau} & 0 \\ 0 & 0 & e^{\lambda_3\tau} \end{bmatrix}, \tag{49}$$

and $\bar{\chi}_0 \equiv (\bar{\xi}_0, \bar{\eta}_0, \bar{\zeta}_0) = \bar{\chi}(0)$. The three components of $\bar{\chi}(\tau)$ are therefore completely decoupled. Transforming back to the original coordinates we have

$$\chi(\tau) = \mathbf{S}\,e^{\Lambda\tau}\mathbf{S}^{-1}\chi_0, \tag{50}$$

where $\chi_0 = \chi(0)$. In the case that $D > 0$, i.e., when λ_1 is real and λ_2 and λ_3 are complex conjugates, the general solution behaves like an underdamped oscillator, as illustrated in Fig. 4. We interpret this solution as a decaying oscillation due to λ_2 and λ_3 superposed with

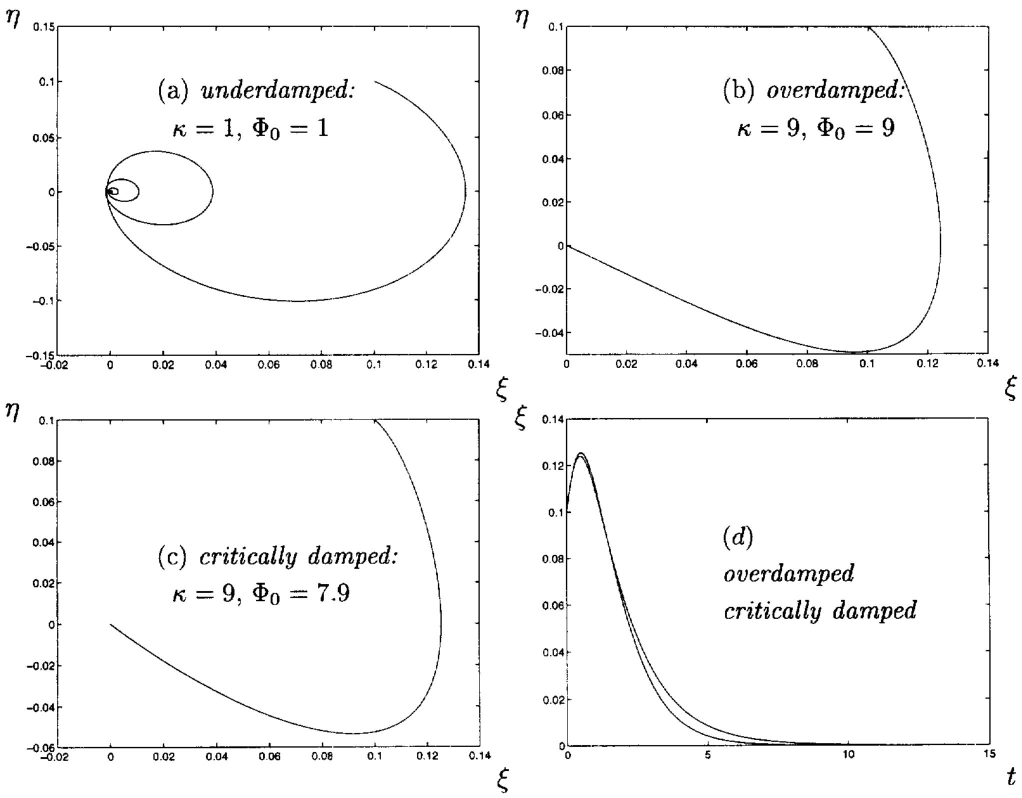

Fig. 4. Examples of the basic types of motion of the unforced, damped linear oscillator corresponding to (a) $D > 0$ (underdamped), (b) $D < 0$ (overdamped), (c) $D = 0$ critically damped, and (d) comparison between the critically damped and overdamped solutions in (b) and (c). The critically damped solution decays faster than the overdamped solution.

a purely exponentially decaying part due to the eigenvalue λ_1. In the case that $D = 0$, $\mathbf{A}$ has only two distinct eigenvalues, λ_1, λ_2, and only two linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$. In this case, the matrix $\mathbf{A}$ can not be diagonalized but it does have a Jordan form. This can be seen through a transformation of coordinates affected by a matrix $\mathbf{M}$ whose first two columns are eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of $\mathbf{A}$ and whose third column is a generalized eigenvector $\boldsymbol{v}_g$, i.e., a solution of

$$(\mathbf{A}-\lambda_2\mathbf{I})^2\boldsymbol{v}_g = 0 \tag{51}$$

for which

$$(\mathbf{A}-\lambda_2\mathbf{I})\boldsymbol{v}_g \neq 0. \tag{52}$$

Such a generalized eigenvector has the form

$$\boldsymbol{v}_g = \begin{bmatrix} 1 \\ \dfrac{-\kappa+\lambda_2^2(2\lambda_2+\kappa)}{3\lambda_2^2+2\kappa\lambda_2+1+2\Phi_0} \\ \dfrac{\lambda_2^4+2(1-\Phi_0)\lambda_2+1+2\Phi_0}{3\lambda_2^2+2\kappa\lambda_2+1+2\Phi_0} \end{bmatrix}. \tag{53}$$

Now, define

$$\mathbf{M} = \begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_g \\ | & | & | \end{bmatrix}, \tag{54}$$

and observe that by introducing the new coordinates $\bar{\boldsymbol{\chi}}(\tau) = (\bar{\xi}, \bar{\eta}, \bar{\zeta})(\tau)$ according to

$$\boldsymbol{\chi}(\tau) = \mathbf{M}\bar{\boldsymbol{\chi}}(\tau) \tag{55}$$

we obtain

$$\frac{\mathrm{d}\bar{\boldsymbol{\chi}}(\tau)}{\mathrm{d}\tau} = \mathbf{J}\bar{\boldsymbol{\chi}}(\tau), \tag{56}$$

where

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}. \tag{57}$$

Clearly $\bar{\xi}(\tau)$ will have the form

$$\bar{\xi}(\tau) = \bar{\xi}_0 e^{\lambda_1\tau}, \tag{58}$$

and is decoupled from the solutions for the other two components $\bar{\eta}(\tau)$ and $\bar{\zeta}(\tau)$. For $\bar{\eta}(\tau)$ and $\bar{\zeta}(\tau)$ we have

$$\bar{\eta}(\tau) = \bar{\zeta}_0 e^{\lambda_2\tau}(\bar{\eta}_0+\bar{\zeta}_0\tau) \tag{59}$$

and

$$\bar{\zeta}(\tau) = \bar{\zeta}_0 e^{\lambda_2 \tau}, \tag{60}$$

respectively. This general solution corresponds to that for a critically damped oscillator as illustrated in Fig. 4. Thus, since the time dependence of the original variables ξ, η, and ζ is given as a linear combination of the solutions for $\bar{\xi}$, $\bar{\eta}$ and $\bar{\zeta}$ according to (55), it may be concluded that the motion is critically damped due to the eigenvalue λ_2 superposed on an exponentially decaying non-oscillating part corresponding to the eigenvalue λ_1.

3.3. *Forced linear oscillations: the magnification factor*

When the external forcing amplitude p is not zero, the general solution of (21) will be the sum of a homogeneous solution, according to the previous section, and a particular solution due to the forcing. Such a particular solution $\boldsymbol{\chi}^*(\tau) = (\xi^*, \eta^*, \zeta^*)(\tau)$ may be represented in the form

$$\boldsymbol{\chi}^*(\tau) = \boldsymbol{\Xi}(\tau) \int_{\tau_0}^{\tau} \boldsymbol{\Xi}^{-1}(s)\mathbf{p}(s)\,\mathrm{d}s, \tag{61}$$

where

$$\mathbf{p}(\tau) = \begin{bmatrix} 0 \\ p\cos\omega\tau \\ 0 \end{bmatrix}, \tag{62}$$

and where $\boldsymbol{\Xi}(\cdot)$ is a fundamental matrix function whose columns consist of linearly independent solutions of the unforced system (32). In the case that $D \neq 0$ it follows that

$$\boldsymbol{\Xi}(s) = \mathbf{S}^{-1} e^{\boldsymbol{\Lambda} s} \tag{63}$$

and when $D = 0$ we find

$$\boldsymbol{\Xi}(s) = \mathbf{M}^{-1} e^{\mathbf{J} s}, \tag{64}$$

where $\mathbf{S}$, $\boldsymbol{\Lambda}$, $\mathbf{M}$ and $\mathbf{J}$ are given by (44), (47), (54) and (57), respectively, and where

$$e^{\mathbf{J}s} = \begin{bmatrix} e^{\lambda_1 s} & 0 & 0 \\ 0 & e^{\lambda_2 s} & s e^{\lambda_2 s} \\ 0 & 0 & e^{\lambda_2 s} \end{bmatrix}. \tag{65}$$

Because of possible resonance behavior, it is of particular interest to determine how the amplitude of the motion $\xi^*(\tau)$ depends on the forcing frequency ω for a given forcing amplitude p. To see this, we obtain from (61) to (65) that in both cases $D \neq 0$ and $D = 0$ the solution $\xi^*(\tau)$ may be written in the form

$$\xi^*(\tau) = -p \sum_{i=1}^{3} \frac{L_i}{\lambda_i^2 + \omega^2} (\lambda_i \sin\omega\tau + \omega\cos\omega\tau), \tag{66}$$

where†

$$L_1 = \frac{\lambda_2 + \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)},$$

$$L_2 = \frac{\lambda_3 + \lambda_1}{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)},$$

$$L_3 = \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}. \tag{67}$$

† The apparent singularities in L_i in the case $D = 0$ when $\lambda_2 = \lambda_3$ are spurious. The summation occurring in (66) as well as (69) and (70) are, in fact, finite.

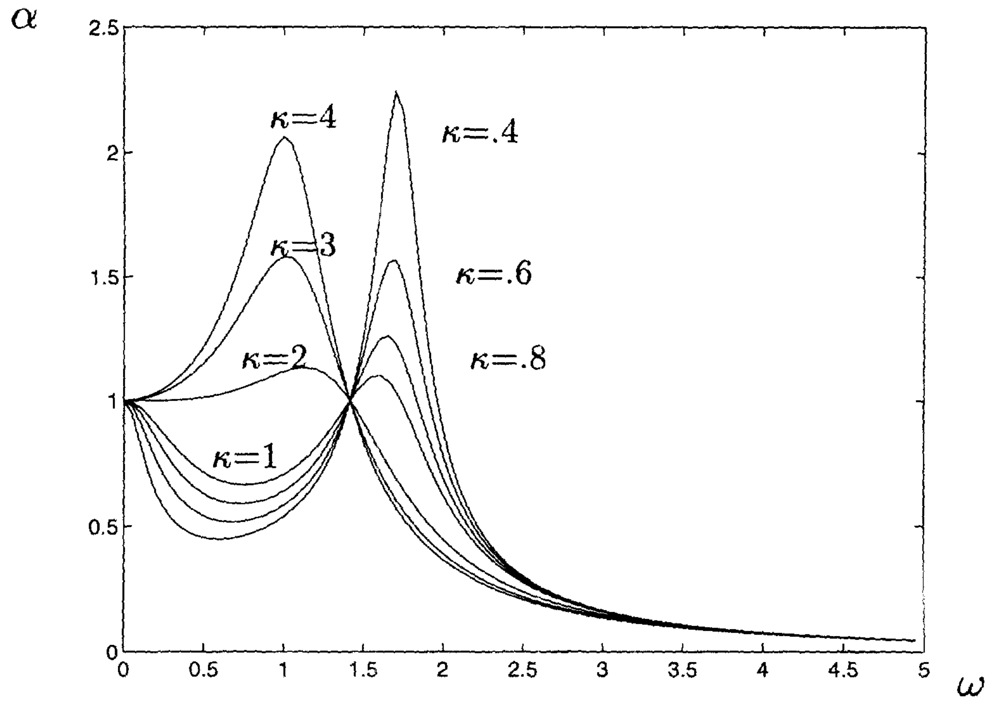

Fig. 5. The magnification factor α vs ω for various values of κ and $\Phi_0 = 1$.

Alternatively, we have

$$\xi^*(\tau) = p\alpha \sin(\omega\tau - \phi), \tag{68}$$

where

$$\alpha = \sqrt{\left(\sum_{i=1}^{3} \frac{L_i\lambda_i}{\lambda_i^2+\omega^2}\right)^2 - \left(\sum_{i=1}^{3} \frac{L_i\omega}{\lambda_i^2+\omega^2}\right)^2} \tag{69}$$

and

$$\phi = \arctan\left(\left(\sum_{i=1}^{3} \frac{L_i\lambda_i}{\lambda_i^2+\omega^2}\right) \Big/ \left(\sum_{i=1}^{3} \frac{L_i\omega}{\lambda_i^2+\omega^2}\right)\right). \tag{70}$$

In Figs 5 and 6 we take $\Phi_0 = 1$ and $\Phi_0 = 2$, respectively, and show the *magnification factor* $\alpha \equiv$ amplitude $|\xi^*(\tau)|/p$ as a function of ω for various values of κ. Note that as κ increases the curves resemble more and more those that would be obtained for the case of classical viscous damping. Figures 7 and 8 show, correspondingly, the phase lag ϕ of $\xi^*(\tau)$ relative to the phase of the forcing function $p\cos\omega t$. Again, as κ increases the curves resemble those that would be obtained for viscous damping. In Section 4 we show that classical viscous damping indeed characterizes the limiting behavior corresponding to history type damping as κ becomes large.

History type damping has the very distinct feature of leading to two resonance frequencies. One of these is clearly the natural frequency $\omega = 1$ of the undamped unforced oscillator. This is the resonance frequency for large κ when the memory of the material is short, in which case it is not surprising that the damping effect may be approximated by the action of a classical viscous damper. It is interesting to observe from Figure 5 and 6 that the resonance frequency can be significantly shifted by changing the parameter κ. All

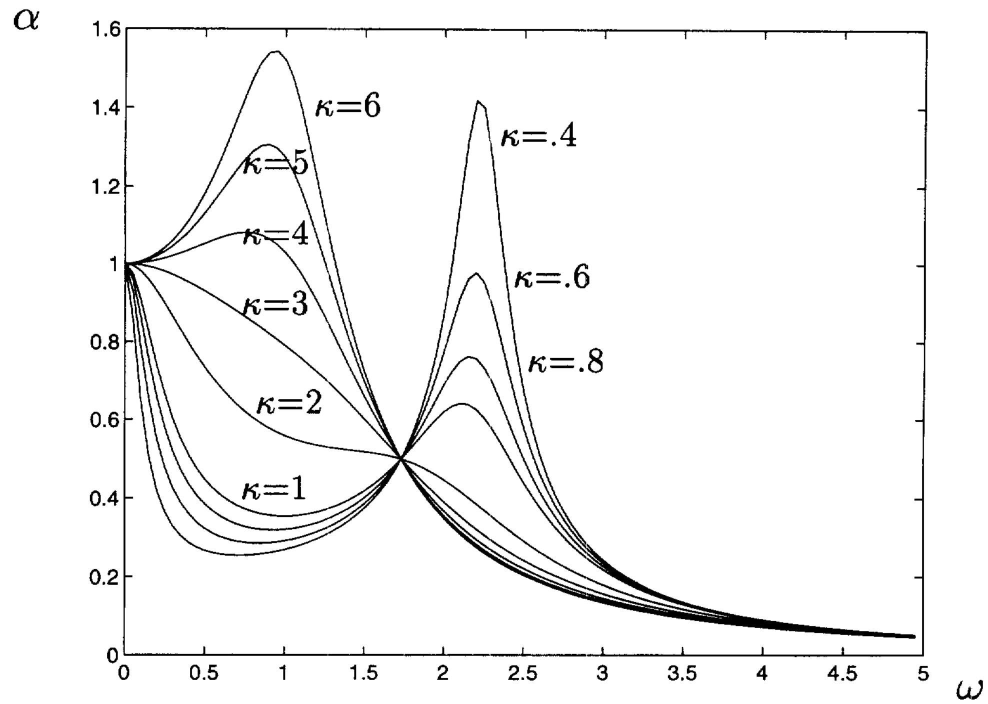

Fig. 6. The magnification factor α vs ω for various values of κ and $\Phi_0 = 2$.

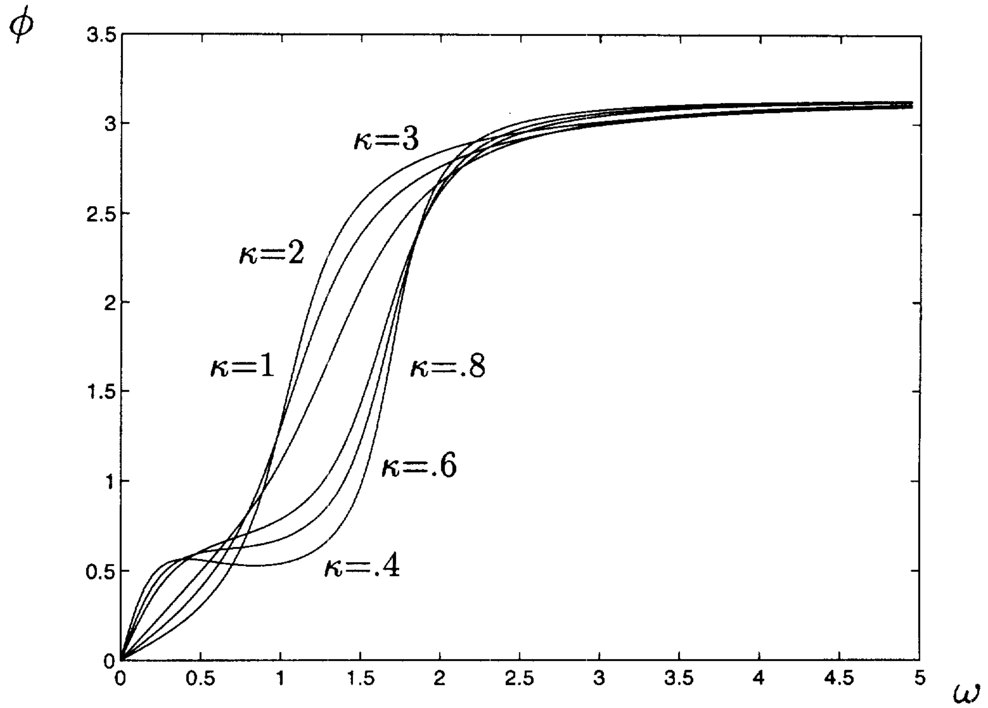

Fig. 7. The phase shift ϕ vs ω for various values of κ and $\Phi_0 = 1$.

other things fixed, this amounts to changing the relaxation time γ of the material, and suggests the use of the device as an active component in a system that requires resonance modulation.

3.4. *Forced linear oscillations: the transmission ratio*

In a vibrating system such as that studied here, the amplitude p of the applied force (scaled) $p\cos\omega t$ is generally not the amplitude of the force that is transmitted to the fixed

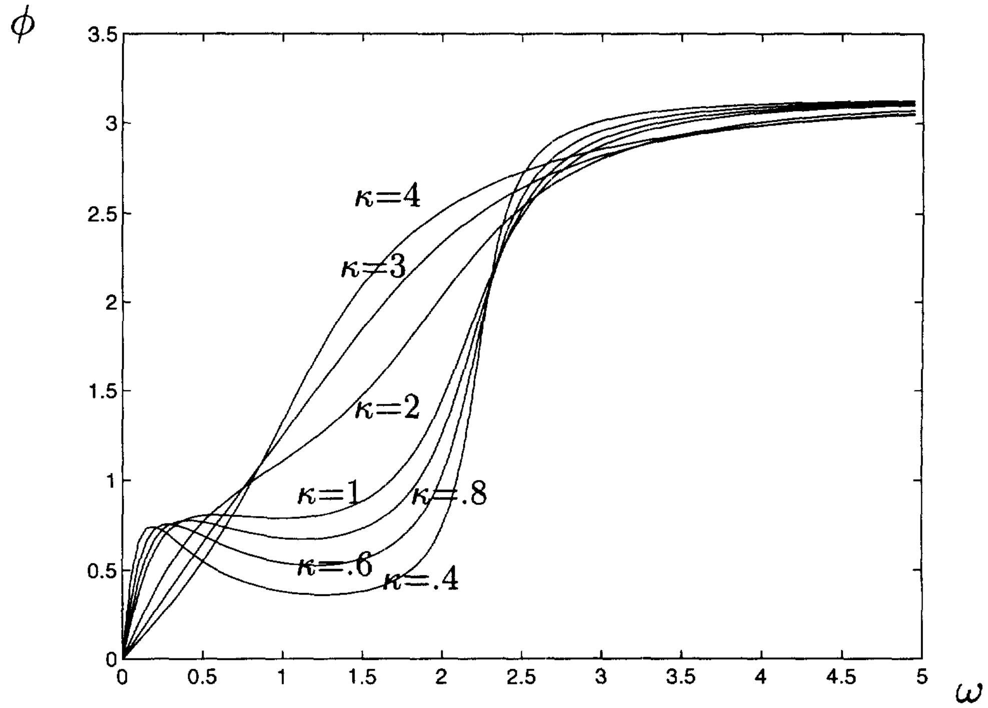

Fig. 8. The phase shift ϕ vs ω for various values of κ and $\Phi_0 = 2$.

foundation. The latter force (scaled according to (21)) is given by $\zeta^*(\tau) - \Phi_0\zeta^*(t)$, and the *transmission ratio*, $\beta \equiv$ amplitude $[\zeta^*(\tau) - \Phi_0\zeta^*(t)]/p$, is an important parameter in the study of vibration isolation (see, e.g., Steidel (1989)).

To illustrate this, consider the case $D \neq 0$ of non-critical damping ($D = 0$ can be handled similarly). In (66) we have recorded $\xi^*(\tau)$, and using (61)–(65) it follows that

$$\Phi_0\zeta^*(\tau) = -p \sum_{i=1}^{3} L_i \frac{\lambda_i^2 + 1}{\lambda_i^2 + \omega^2}(\lambda_i \sin \omega\tau + \omega \cos \omega\tau), \tag{71}$$

where L_1, L_2 and L_3 are as defined in (67). Thus, we show in Figs 9 and 10 the transmission ratio for the two choices $\Phi_0 = 1$ and $\Phi_0 = 2$, respectively, and for the various choices of κ. Note that for certain combinations of κ and Φ_0, the transmitted force is smaller than the amplitude of the forcing for all frequencies $\omega < \sim 1.4$. This is in contrast to the case of an oscillator with classical viscous damping where for all frequencies below a certain level the transmission ratio $\beta > 1$ regardless of the value of the viscosity (see, e.g., Steidel (1989)). Notice also, by comparing Figs 5 and 6, that for a given value of κ an increase in Φ_0 will lead to a decrease in transmissibility for all values of ω.

An important distinction between classical viscous damping and the present type of history damping is that for the latter, vibration isolation may be obtained, (i.e., $\beta < 1$) for forcing frequencies that are less than the natural frequency of the unforced, undamped oscillator. As is well known, for linear oscillators with viscous damping $\beta < 1$ is only possible when the forcing frequency is greater than $\sqrt{2}$ times the natural frequency. This distinct feature can be used in the design of an adaptive vibration damper where the transmissibility can be controlled by adjusting the material parameters.

4. DAMPING EFFECTS IN THE LIMITS OF SHORT AND LONG MEMORY

The damping effects of short and long memory correspond to the respective cases of when the natural time constant γ satisfies either $\gamma \ll 1$ or $\gamma \gg 1$. Because of (35), these cases correspond, respectively, to $\kappa \gg 1$ and $\kappa \ll 1$. The parameter $\kappa = 1/\gamma\omega_0$ enters the dynamical

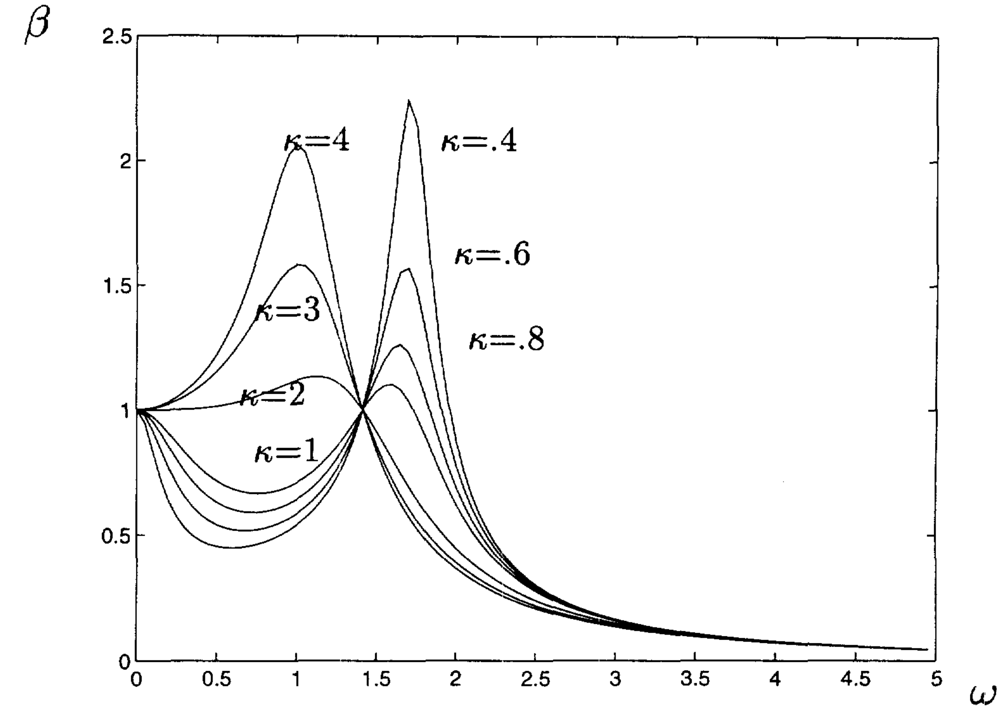

Fig. 9. The transmission ratio β vs ω for various values of κ and $\Phi_0 = 1$.

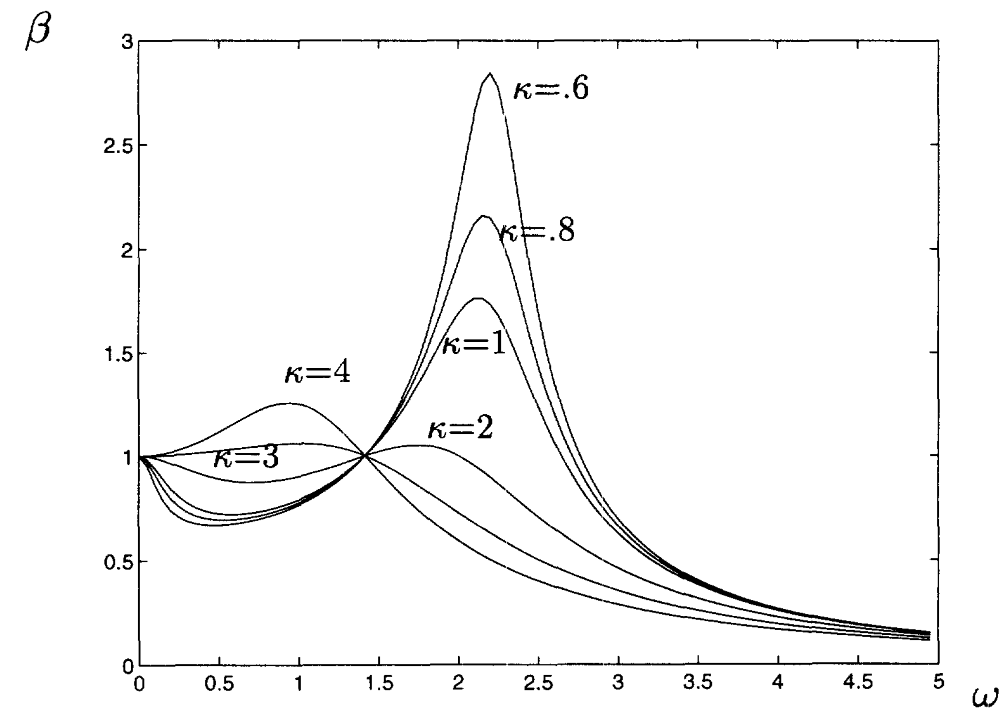

Fig. 10. The transmission ratio β vs ω for various values of κ and $\Phi_0 = 2$.

equation (21) only through the third equation, and for our purposes, here, we shall eliminate the first equation and write this system in the form

$$\begin{aligned} \xi''(\tau) &= -\xi(\tau)+\Phi_0\zeta(\tau)+p\cos\omega\tau, \\ \zeta'(\tau) &= -\kappa\zeta(\tau)-2\xi'(\tau), \end{aligned} \tag{72}$$

where the superposed "′" denotes differentiation with respect to τ. Equivalently, we write $(72)_2$ as

$$\zeta(\tau) = \zeta(0)e^{-\kappa\tau} - 2\int_0^\tau e^{-\kappa s}\xi'(\tau-s)\,\mathrm{d}s. \tag{73}$$

It follows from integrations by parts that

$$\zeta(\tau) = \zeta(0)e^{-\kappa\tau} - \frac{2}{\kappa}\xi'(\tau) + \frac{2}{\kappa}\left[e^{-\kappa\tau}\xi'(0) + \int_0^\tau e^{-\kappa s}\xi''(\tau-s)\,\mathrm{d}s\right]. \tag{74}$$

It also follows from a different integration by parts of (73) and substitution into $(72)_2$, that

$$\zeta'(\tau) = -\kappa\zeta(0)e^{-\kappa\tau} - 2\xi'(\tau) + 2\kappa\xi(\tau) - 2\kappa\left[e^{-\kappa\tau}\xi(0) + \kappa\int_0^\tau e^{-\kappa s}\xi(\tau-s)\,\mathrm{d}s\right]. \tag{75}$$

Thus, for short memory when $\kappa \gg 1$ we see from (74) that

$$\zeta(\tau) = -\frac{2}{\kappa}\xi'(\tau) + \mathrm{O}\left(\frac{1}{\kappa^2}\right), \tag{76}$$

and in this case the system (72) may be replaced by the single equation

$$\xi''(\tau) + \frac{2\Phi_0}{\kappa}\xi'(\tau) + \xi(\tau) = p\cos\omega\tau, \tag{77}$$

which is characteristic of a classical spring-viscous damper dynamical system with a small viscosity, $\mu = 2\Phi_0/\kappa$. When $\mu = 0$ (i.e., $\kappa = \infty$) the natural frequency equals 1, which corresponds to the first natural frequency that is shown in Figs 5 and 6.

For long memory when $\kappa \ll 1$ we see from (75) that

$$\zeta'(\tau) = -\kappa\zeta(0) - 2\xi'(\tau) + 2\kappa(\xi(\tau) - \xi(0)) + \mathrm{O}(\kappa^2). \tag{78}$$

Clearly, for the limiting situation $\kappa = 0$ we have $\zeta(\tau) = \zeta(0) - 2(\xi(\tau) - \xi(0))$, and the system (72) may be replaced by

$$\xi''(\tau) + (1 + 2\Phi_0)\xi(\tau) = \Phi_0(\zeta(0) + 2\xi(0)) + p\cos\omega\tau, \tag{79}$$

which is characteristic of a classical undamped linear oscillator with natural frequency equal to $\sqrt{1+2\Phi_0}$. This corresponds to the second natural frequency that is shown in Figs 5 and 6.

5. CLOSING REMARKS—COMPARISON TO VISCOUS DAMPING

Linear oscillators that are damped through the use of materials with memory exhibit several characteristics that are qualitatively different from those of oscillators with classical viscous damping. These characteristics may have important technological applications. A striking difference is seen in the respective magnification factors (cf. Figs 5 and 6) of the two types of damping. For history type damping, special combinations of the two material parameters, which corresponds to large relaxation times, will lead to a new resonance frequency that is not present for oscillators with classical viscous damping.

The transmission ratio (cf. Figs 9 and 10) for the oscillator and history type damping exhibits the same resonance phenomena as the magnification factor. In addition, it is seen that the transmission ratio may be less than 1, i.e., it is possible to have vibration isolation for forcing frequencies both below and above the natural frequency of the unforced oscillator. This should be compared to the behavior of linear oscillators with classical viscous damping where the transmission ratio is less than 1 only for forcing frequencies sufficiently greater than the natural frequency of the oscillator. In fact, if ω_n denotes the natural frequency in this case, the forcing frequency must be greater than $\sqrt{2}\omega_n$ for effective vibration isolation.

For the case of the material with memory, it is possible that the governing parameters of the damping, i.e., the relaxation time γ and the relaxation modulus Φ_0, could be controlled by the temperature or electric field (for electro-rheologically active materials). In general, then, this device has potential as an active component in vibration control.

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